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ON GENERALIZED AMENABILITY

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Abstract. There is a word metric d on countably generated free group Γ such that (Γ, d) does not admit a coarse uniform imbedding into a Hilbert space.

§1 Introduction

A discrete countable group G is called *amenable* if there exists a left invariant *mean* on G, i.e. a positive finitely additive, finite measure μ. Clearly, that $\mu(g) = 0$ for all $g \in G$. Equivalently, a group G is amenable if its natural action on the Stone-Čech compactification βG admits an invariant measure. In [Gr] M. Gromov introduced the notion of an *a*-T- *menable* group as a group G which admits a proper isometric action on the Hilbert space l_2.

The Novikov higher signature conjecture was known for some classes of amenable groups for many years. Recently Higson and Kasparov [H-K] proved it for all amenable groups and for a-T-menable groups. Then G. Yu [Y] proved it for more general class of groups, we call it *Y-amenable* groups. A group G is called Y-amenable if it admits a coarsely uniform embedding as defined in [Gr] into a Hilbert space.

In the case of genuine amenability there is the Folner Criterion [Fo],[Gr] which allows to establish amenability of a group in terms of the growth function of an exhausting family of compact sets in a group. In [Y] Yu introduced his Property A (we do not define it here), which serves as a distant analog of Folner property. After analyzing the Property A Higson and Roe [H-R] introduced a new notion of amenability.

Definition. *A discrete countable group G is called Higson-Roe amenable if its action on the Stone-Čech compactification βG is topologically amenable.*

An action of G on a compact space X is *topologically amenable* [A-D-R] if there is a sequence of continuous maps $b^n : X \to P(G)$ to the space of probability measures on G such that for every $g \in G$, $\lim_{n\to\infty} \sup_{x\in X} \|gb^n_x - b^n_{gx}\|_1 = 0$. Here a measure $b^n_x = b^n(x)$ is considered as a function $b^n_x : G \to [0,1]$ and $\| \ \|_1$ is the l_1-norm.

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Assertion 1. *A discrete countable group G is Higson-Roe amenable if it admits a topologically amenable action on some compact metrizable space X.*

Proof. The proof in one direction is given in Proposition 2.3 of [H-R]. The other implication follows from countability of G and the Schepin Spectral Theorem [Ch]. $\square$

We note that the trivial action of the classiscal amenable groups on a one-point space is topologically amenable. Also all hyperbolic groups are acting on their Gromov boundaries topologically amenable [Ad], [A-D-R]. Still there is no example of a countable group which is not Higson-Roe amenable. In this paper we present an example of countable group which is not Y-amenable.

§2 Coarsely uniform embeddings

A map $f : X \to Y$ between metric spaces is called a *coarsely uniform embedding* if there are functions $\rho_1, \rho_2; [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} \rho_i(t) = \infty$ such that
$\rho_1(d_X(x,x')) \le d_Y(f(x), f(x')) \le \rho_2(d_X(x,x'))$ for all $x, x' \in X$.

1. Higson-Roe-Yu Embedding Theorem. The following theorem is due to Higson-Roe and Yu [H-R],[Y].

Theorem 1. *A finitely generated Higson-Roe amenable group G admits a coarsely uniform embedding into the Hilbert space for a word metric on G.*

Every set of generators S of a group G defines a word metric d_S on G. If a group is finitely generated we assume that S is finite. Any two such metrics generated by two finite sets are quazi-isometric. The following fact is well-known.

Assertion 2. *Let Γ be a finitely generated subgroup of a finitely generated group G, then the inclusion is a coarsely uniform embedding.*

Proof. Let S be a set of generators of Γ and let T be a set of generators of G. Without loss of generality we may assume that $S \subset T$. Then $d_T(x,y) = \|x^{-1}y\|_T \le \|x^{-1}y\|_S = d_S(x,y)$. Thus, $\rho_2(t) = t$. We define $\rho_1(t) = \min\{\|w\|_T \mid w \in \Gamma, \|w\|_S \ge t\}$. Assume that ρ_1 is bounded. Then there are a constant R and a sequence $w_i \in \Gamma$ with $\|w_i\|_S \ge i$ and $\|w_i\|_T \le R$. This contradicts with the fact that a R-ball in G is finite. Note that $\rho_1(d_S(x,y)) \le \|x^{-1}y\|_T = d_T(x,y)$. $\square$

2. Modified Enflo's Metric Spaces. We define metric spaces M_n which are adaptations for asymptotic geometry of Enflo's spaces [En]. Let $N_n = \{0, 1, 2, \dots, 2^{n+1}-1\}$ with metric $|x-y| \mod 2^{n+1}$. We define $M_n = (N_n)^{2n^n}$ as the product of $2n^n$ copies of N_n with the metric $d(a,b) = \max_i\{|a_i - b_i| \mod 2^{n+1}\}$ where $a = \{a_i\}$ and $b = \{b_i\}$.

A pair of points (a,b) in M_n is called an m-segment if the coordinates of a and b are different in exactly $2n^{n-m}$ positions and $|a_i - b_i| = 2^m$ if $a_i \ne b_i$.

Proposition 1. *For any two m-segments (a,b) and (a',b') in M_n there is an isometry $h : M_n \to M_n$ with $h(a) = a'$ and $h(b) = b'$ such that h takes k-segments to k-segments for any k.*

Proof. First we consider a permutation $\sigma : \{1, 2, \dots, 2n^n\} \to \{1, 2, \dots, 2n^n\}$ which establishes a bijection between coordinate spaces for which $a_i = b_i$ and $a'_i = b'_i$. Then for every

i we consider an isometry $h_i : N_n \to N_n$ taking $(a_{\sigma(i)}, b_{\sigma(i)})$ to (a'_i, b'_i). Such an isometry exists, since either $|a_{\sigma(i)} - b_{\sigma(i)}| = 2^m = |a'_i - b'_i|$ or $|a_{\sigma(i)} - b_{\sigma(i)}| = 0 = |a'_i - b'_i|$. The family $\{h_i\}$ defines an isometry $\bar{h} : M_n \to M_n$. We define $h = \bar{h} \circ \bar{\sigma}$ where $\bar{\sigma} : M_n \to M_n$ is defined by the formula $\bar{\sigma}(x_1, \dots, x_{2n^n}) = (x_{\sigma(1)}, \dots, x_{\sigma(2n^n)})$. Then $(h(a))_i = (\bar{h} \circ \bar{\sigma}(a))_i = h_i(\bar{\sigma}(a))_i) = h_i(a_{\sigma(i)}) = a'_i$. Thus, $h(a) = a'$. Similarly, one can check that $h(b) = b'$. $\square$

Following Enflo [En], by a *double n-simplex* in a space M we call a set $2n+2$ points $D_n = \{u_1, \dots, u_{n+1}, v_1, \dots, v_{n+1}\}$, $u_i, v_j \in M$. Pairs (u_i, u_j) and (v_i, v_j) are called *edges* of D_n and pairs $(u - i, v_j)$ are called *connecting lines.*

Proposition 2. *For any m, $1 \le m < n$, there exists a double $(n-1)$-simplex $D^m_{n-1} \subset M_n$ such that all edges are m-segments and all connecting lines are $(m-1)$-segments.*

Proof. Let $I_m = \{1, \dots, n^{n-m+1}\}$. Let $J^m_1, \dots, J^M_n$ be the partition of I_M in n equal parts $J^m_1 = \{1, \dots, n^{n-m}\}$, $J^m + 2 = \{n^{n-m} + 1, \dots, 2n^{n-m}\}$, ..., $J^M_n = \{(n-1)n^{n-m} + 1, \dots, n^{n-m+1}\}$. We define $u_k, v_k \in M_n$ as follows:

$$(u_k)_i = \begin{cases} 2^m, & if\ i \in J^m_k; \\ 2^{m-1}, & if\ i \in I_m + n^{n-m+1}; \\ 0, & otherwise. \end{cases}$$

and

$$(v_k)_i = \begin{cases} 2^m & if\ i \in J^m_k + n^{n-m+1}; \\ 2^{m-1} & if\ i \in I_m; \\ 0 & otherwise. \end{cases}$$

Since u_k and u_l for $k \ne l$ differ at $2n^{n-m}$ positions and $|(u_k)_i - (u_l)_i| = 2^m$ at those positions, all u-edges in the corresponding double $(n-1)$-simplex are m-segments. Similarly, all v-edges are m-segments. Since u_k and v_l are distinct in $2n^{n-m+1}$ coordinates with $|(u_k)_i - (v_l)_i| = 2^{m-1}$, every connecting line (u_k, v_l) is an $(m-1)$-segment. $\square$

3. Obstruction to Embedding The following proposition is well-known, it can be extracted from [En].

Proposition 3. *For every double n-simplex in the Hilbert space the inequality $\Sigma c^2_\alpha \ge \Sigma s^2_\beta$ holds where c_α runs through the length of connecting lines and s_β runs through the length of edges.*

Proof. First we proof this inequality for a double simplex in the real line. The equality
$\Sigma_{1 \le k,l \le n}(u_k - v_l)^2 - \Sigma_{1 \le k < l \le n}(u_k - u_l)^2 - \Sigma_{1 \le k < l \le n}(v_k - v_l)^2$
$= (\Sigma_{1 \le k \le n} u_k - \Sigma_{1 \le l \le n} b_l)^2$
implies the inequality
$\Sigma_{1 \le k,l \le n}(u_k - v_l)^2 \ge \Sigma_{1 \le k < l \le n}(u_k - u_l)^2 + \Sigma_{1 \le k < l \le n}(v_k - v_l)^2$
which is exactly the inequality $\Sigma c^2_\alpha \ge \Sigma s^2_\beta$.

Since $\|u_k - u_l\|^2 = \Sigma_i((u_k)_i - (v_l)_i)^2$, we obtain the required inequality by adding up corresponding inequalities for i-th coordinates. $\square$

Theorem 2. *Assume that a metric space X contains an isometric copies of M_n for all n Then X cannot be coarsely uniformly embedded in the Hilbert space.*

Proof. Assume the contrary. Let $f : X \to l_2$ be a coarsely uniform embedding. let ρ_1 and ρ_2 be corresponding functions. Since $\rho_1 \to \infty$, there is m such that $\rho_1(2^m) > 2\sqrt{e}\rho_2(1)$. We consider a double $(n-1)$-simplex $D^m_{n-1} \subset M_n \subset X$ for any $n > m$. Denote by $\bar{f}((a,b)) = \|f(a) - f(b)\|$. Then Proposition 3 implies the inequality

$$\Sigma_{c\in C(D^m_{n-1})}\bar{f}(c)^2 \geq \Sigma_{s\in E(D^m_{n-1})}\bar{f}(s).$$

Here $C(D^m_{n-1})$ denotes the set of all connecting lines and $E(D^m_{n-1})$ denotes the set of all edges.

Let $\mathcal{D}$ be the set of all double $(n-1)$ simplices in M_n isomorphic to D^m_{n-1}. Then

$$\Sigma_{c\in C(D),D\in\mathcal{D}}\bar{f}(c)^2 \geq \Sigma_{s\in E(D),D\in\mathcal{D}}\bar{f}(s)^2.$$

This inequality can be written as

$$\Sigma_{c\in C(D),D\in\mathcal{D}}\frac{\bar{f}(c)^2}{n^2 card(\mathcal{D})} \geq \Sigma_{s\in E(D),D\in\mathcal{D}}\frac{\bar{f}(s)^2}{n^2 card(\mathcal{D})} = \frac{n-1}{n}\Sigma_{s\in E(D),D\in\mathcal{D}}\frac{\bar{f}(s)^2}{n(n-1)card(\mathcal{D})}.$$

Let S_m denote the set of all m-segments in M_n. By the Proposition 1 all m-segments in M_n are equal. It means that every m-segment c is a connecting line for the same number of double simplices from $\mathcal{D}$ and every $m-1$-segment is an edge of the same number of double simplices from $\mathcal{D}$. Since the number of connecting edges in a double $(n-1)$-simplex is n, the expression $\Sigma_{c\in C(D),D\in\mathcal{D}}\frac{\bar{f}(c)^2}{n^2 card(\mathcal{D})}$ is the arithmetic mean. Because of symmetry the arithmetic mean $\Sigma_{c\in C(D),D\in\mathcal{D}}\frac{\bar{f}(c)^2}{n^2 card(\mathcal{D})}$ can be computed as $\Sigma_{c\in S_{m-1}}\frac{\bar{f}(c)^2}{card(S_{m-1})}$. By a similar reason the arithmetic mean $\Sigma_{s\in E(D),D\in\mathcal{D}}\frac{\bar{f}(s)^2}{n(n-1)card(\mathcal{D})}$ can be computed as $\Sigma_{c\in S_m}\frac{\bar{f}(c)^2}{card(S_m)}$. Thus, we have an inequality

$$(1+\tfrac{1}{n-1})\bar{g}_{m-1} \geq \bar{g}_m \text{ where } \bar{g}_k = \Sigma_{c\in S_k}\frac{\bar{f}(c)^2}{card(S_k)}.$$

Iterate this inequality to obtain the following
$(1+\frac{1}{n-1})^{n-1}\bar{g}_o \geq \bar{g}_{n-1}$. Hence $e\bar{g}_0 \geq \bar{g}_{n-1}$. Then
$\sqrt{e}\rho_2(1) \geq \sqrt{e}\sup_{c\in S_0}\bar{f}(c) \geq \inf_{c\in S_{n-1}}\bar{f}(c) \geq \rho_1(2^{n-1}) \geq \rho_1(2^m) > 2\sqrt{e}\rho_2(1)$.
The contradiction completes the proof. $\square$

4. A group which is not Y-amenable.

For every modified Enflo's space M_n we consider the graph G_n whose vertices are points of M_n and two vertices a and b are joint by an edge if and only if $d(a,b) = 1$ in M_n. Note that G_n is connected. Let G be an infinite wedge of all G_n. We define a path

metric on G such that any two vertices joined by an edge are on distance one. We define a countable infinitely generated group Γ as follows. Fix an orientation on all edges of G. Then all edges of G are the generators of Γ and all loops are the relations.

Theorem 3. *The group Γ is not Y-amenable.*

Proof. Fix a metric on Γ defined by the above set of generators, then G is isometrically imbedded in Γ. By Theorem 2 Γ does not admit a coarsely uniform embedding into the Hilbert space. $\square$

It can be shown that the group Γ is infact a free group generated by edges of a maximal tree in G.

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